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Principal investigator and contractor: Prof.F.Cap

Institute of Theoretical Physics

University of Innsbruck, Innsbruck, Austria

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ON THE SOLUTION OF THE HEAVY ASYMMETRIC GYROSCOPE, USING THE PROPERTIES OF THE LIE OPERATOR.

by

F.Cap and A.Schett

ABSTRACT

Using Lie series the solution of the equation describing the heavy asymmetric gyroscope is presented in the forms:

A) Solution = Solution(heavy, symmetric) + contributions from asymmetry.

B) Solution = Solution (symmetric, forcefree) + contributions from asymmetry and forces.

C) Solution = Solution (asymmetric, forcefree) + contributions from forces.

The 1-st term on the right side is presented in global form and the 2-nd term, an integral term, can be computed by iterations.

Which of the aforementioned representations is most suitable, depends on the problem considered.

1. Introduction

We consider an asymmetric gyroscope with several torque producing forces. A spinning satellite is essentially such a gyroscope; the torque may result in a change in satellite orientation that affects the thermal balance, the operation of solar cells and various scientific measurements.

By means of an operator we can represent the solution of the heavy asymmetric gyroscope such, that the contributions of the different torques appear separately. In other words, using a splitting up procedure of the afore-mentioned operator we can represent the solution in the form, e.g.:

$$\Omega = \Omega_{\substack{\text{global}\\ \text{symmetric}\\ \text{forcefree}}} + \sum_{\alpha=0}^{\infty} \int_{t_0}^{t} \sum_{i} M_i f_\alpha(\tau, M_i, \Omega_{gl})\, d\tau + \sum_{\alpha=0}^{\infty} \int_{t_0}^{t} A\, f_{1\alpha}(\tau, M_i, \Omega_{gl})\, d\tau$$

where $\Omega_{global} = \Omega_{gl}$ indicates that this function is represented in a global form.

The torques M_i ($i = 1, 2, 3 \ldots.$) appearing in the integral terms usually differ by their order of magnitude. For a given degree of accuracy, therefore, the number of summation terms α to be computed depends on the integral considered. The afore mentioned solution representation renders it possible to compute the single integral terms as accurate as necessary irrespective of the other integral terms.

Especially we shall present the solution of the equation descibing the heavy asymmetric gyroscope in the forms:

A) Solution = Solution (heavy, symmetric) + contributions from asymmetry

The 1-st term is exactly known, i.e., there exists a global solution representation. The 2-nd term can be calculated by iterations.

B) Solution = Solution (symmetric, forcefree) + contributions from asymmetry and forces.

The 1-st term is again exactly known. The 2-nd term can be split up into several additive integral terms:

a) a term containing the contributions of asymmetry; this term vanishes if the satellite (gyroscope) is symmetric.

b) additive integral terms containing the torques M_i $(i=1,2, ..)$

$$\sum_{\alpha} \int \sum_{i} M_i f_{\alpha}(\tau, M_i, \Omega_{gl}) d\tau,$$

i.e. these terms vanish if $M_i = 0$

C) Solution = Solution (asymmetric, forcefree) + contributions from forces.

For the first term a global solution representation exists and the second term can be computed by iterations.

The problem which of the afore-mentioned solution representations is most suitable, depends on the problem considered. Generally, one can say, that it is advantageous to put the main contribution of the solution in the 1-st term and perturbations in the remaining terms.

2. Solution of the equation describing the heavy asymmetric gyroscope

Using a reference frame (ξ, η, ζ) fixed with respect to the body the equation of the heavy asymmetric gyroscope reads

$$\begin{aligned}
M_1 &= I_1\dot{\Omega}_1 + (I_2 - I_3)\Omega_2\Omega_3 \\
M_2 &= I_2\dot{\Omega}_2 + (I_3 - I_1)\Omega_1\Omega_3 \\
M_3 &= I_3\dot{\Omega}_3 + (I_1 - I_2)\Omega_2\Omega_1
\end{aligned} \qquad (1)$$

where I_1, I_2, I_3 are the moments of inertia, Ω_1, Ω_2, Ω_3 are the angular velocities, $\dot{\Omega}_1$, $\dot{\Omega}_2$, $\dot{\Omega}_3$, are the angular accelerations and M_1, M_2, M_3 are the torques in the reference frame (1,2,3). In Eq.(1) we have substituted ξ, η, ζ, by 1, 2, 3 .

Using the well-known relations

$$\begin{aligned}
\Omega_1 &= \dot{\varphi}\sin\chi\sin\vartheta + \dot{\vartheta}\cos\chi \\
\Omega_2 &= \dot{\varphi}\cos\chi\sin\vartheta - \dot{\vartheta}\sin\chi \\
\Omega_3 &= \dot{\varphi}\cos\vartheta + \dot{\chi} ,
\end{aligned} \qquad (2)$$

where φ, ϑ, χ are the Eulerian angles defined as follows

$$\sphericalangle\begin{matrix} z \text{ axis} \\ 3 \text{ axis}\end{matrix} : \vartheta \ ; \quad \sphericalangle\begin{matrix} x' \text{ axis} \\ x \text{ axis}\end{matrix} : \varphi \ ; \quad \sphericalangle\begin{matrix} x' \text{ axis} \\ 1 \text{ axis}\end{matrix} : \chi$$

where x,y,z are the axis fixed with respect to the space, x' indicates the nodal line of the two planes (xy) and (12).

Differentiating Ω_i $(i = 1,2,3)$ in Eq,(2) with respect to t (time) and substituting these quantities into Eq.(1) we obtain the following equations

$$\ddot{\varphi} + \dot{\varphi}\dot{\chi}h_{11} + \dot{\varphi}\dot{\vartheta}h_{12} + \ddot{\vartheta}h_{13} + \dot{\vartheta}\dot{\chi}h_{14} + \dot{\varphi}^2h_{15} + \dot{\vartheta}\dot{\varphi}h_{16} + $$
$$+ \dot{\varphi}\dot{\chi}h_{17} + \dot{\vartheta}\dot{\chi}h_{18} = S_1(\vartheta, \chi, \varphi) \qquad (3)$$

$$\ddot{\vartheta} + \ddot{\varphi}h_{21} + \dot{\chi}\dot{\varphi}h_{22} + \dot{\varphi}\dot{\vartheta}h_{23} + \dot{\vartheta}\dot{\chi}h_{24} + \dot{\varphi}^2h_{25} + \dot{\vartheta}\dot{\varphi}h_{26} + $$
$$+ \dot{\varphi}\dot{\chi}h_{27} + \dot{\vartheta}\dot{\chi}h_{28} = S_2(\vartheta, \chi, \varphi) \qquad (4)$$

$$\ddot{\chi} + \ddot{\varphi} h_{31} + \dot{\varphi}\dot{\vartheta} h_{32} + \dot{\varphi}^2 h_{33} + \dot{\varphi}\dot{\vartheta} h_{34} + \dot{\varphi}\dot{\vartheta} h_{35} + \dot{\vartheta}^2 h_{26} = S_3(\vartheta, \chi, \varphi) \,, \tag{5}$$

where $h_{ij} = h_{ij}(\vartheta, \varphi, \chi)$; $i, j = 1, 2, \ldots.$

$S_i = S_i(\vartheta, \varphi, \chi)$; $i = 1, 2, 3$

$$h_{11} = \frac{\cos\chi}{\sin\chi} \qquad h_{15} = \alpha_1 \frac{\cos\chi\cos\vartheta}{\sin\chi}$$

$$h_{12} = \left(\frac{\sin\vartheta}{\cos\vartheta}\right)^{-1} \qquad h_{16} = -\alpha_1 \frac{\cos\vartheta}{\sin\vartheta}$$

$$h_{13} = \frac{\cos\chi}{\sin\chi\sin\vartheta} \qquad h_{17} = \alpha_1 \frac{\cos\chi}{\sin\chi} \tag{6}$$

$$h_{14} = \frac{-1}{\sin\vartheta} \qquad h_{18} = -\frac{\alpha_1}{\sin\vartheta}$$

$$S_1 = \frac{M_1}{I_1} \qquad \alpha_1 = \frac{I_3 - I_2}{I_1} \tag{7}$$

$$h_{21} = -\frac{\cos\chi\sin\vartheta}{\sin\chi} \qquad h_{24} = \frac{\cos\chi}{\sin\chi}$$

$$h_{22} = \sin\vartheta \qquad h_{25} = -\alpha_2 \sin\vartheta\cos\vartheta$$

$$h_{23} = -\frac{\cos\chi\cos\vartheta}{\sin\chi} \qquad h_{26} = -\frac{\cos\vartheta\cos\chi}{\sin\chi}\alpha_2 \tag{8}$$

$$h_{27} = \alpha_2 \sin\vartheta \qquad h_{28} = -\frac{\cos\chi}{\sin\chi}$$

$$S_2 = \frac{M_2}{I_2} \qquad \alpha_2 = \frac{I_2 - I_1}{I_2} \tag{9}$$

$$h_{31} = \cos\vartheta \qquad h_{32} = -\sin\vartheta$$

$$h_{33} = \alpha_3 \sin\chi\cos\chi\sin^2\vartheta$$

$$h_{34} = \alpha_3 \cos^2\chi\sin\vartheta \qquad h_{35} = -\alpha_3 \sin^2\chi\sin\vartheta \tag{10}$$

$$h_{36} = -\alpha_3 \cos\chi\sin\chi$$

$$S_3 = \frac{M_3}{I_3} \qquad \alpha_3 = \frac{I_2 - I_1}{I_3} \tag{11}$$

Inserting $\ddot{\vartheta}$ from Eq.(4) into Eq.(3) we obtain

$$\ddot{\varphi} + \dot{\varphi}\dot{\chi}q_{11} + \dot{\varphi}\dot{\vartheta}q_{12} + \dot{\vartheta}\dot{\chi}q_{13} + \dot{\vartheta}^2 q_{14} + S_1 q_{15} + S_2 q_{16} = 0, \qquad (12)$$

where

$$\begin{aligned}
q_{11} &= \frac{h_{11} + h_{17} + h_{22}h_{13} - h_{13}h_{27}}{1 - h_{21}h_{13}} \\
q_{12} &= \frac{h_{12} + h_{16} - h_{23}h_{13} - h_{26}h_{13}}{1 - h_{21}h_{13}} \\
q_{13} &= \frac{h_{24} + h_{18} - h_{24}h_{13} - h_{28}h_{13}}{1 - h_{21}h_{13}} \\
q_{15} &= \frac{h_{13}}{1 - h_{21}h_{13}} \; ; \quad q_{14} = \frac{h_{15} - h_{25}h_{13}}{1 + h_{21}h_{13}} \\
q_{16} &= - \frac{1}{1 + h_{21}h_{13}}
\end{aligned} \qquad (13)$$

Inserting φ from Eq.(3) into Eq.(4) one obtains

$$\ddot{\vartheta} + \dot{\chi}\dot{\varphi}q_{21} + \dot{\varphi}\dot{\vartheta}q_{22} + \dot{\vartheta}\dot{\chi}q_{23} + \dot{\varphi}^2 q_{24} + S_1 q_{25} + S_2 q_{26} = 0 , \qquad (14)$$

where

$$\begin{aligned}
q_{21} &= \frac{h_{22} + h_{27} - h_{11}h_{21} - h_{14}h_{21}}{1 - h_{13}h_{21}} \\
q_{22} &= \frac{h_{33} + h_{26} - h_{12}h_{21} - h_{16}h_{21}}{1 - h_{13}h_{21}} \\
q_{23} &= \frac{h_{24} + h_{28} - h_{14}h_{21} - h_{18}h_{21}}{1 - h_{13}h_{21}} \\
q_{24} &= \frac{h_{25} - h_{15}h_{21}}{1 - h_{13}h_{21}} \\
q_{25} &= \frac{h_{21}}{1 - h_{13}h_{21}} \qquad q_{26} = - \frac{1}{1 - h_{13}h_{21}}
\end{aligned} \qquad (15)$$

Substituting $\ddot{\varphi}$ in Eq.(5) by Eq.(12) we obtain

$$\ddot{\chi} + \dot{\varphi}\dot{\vartheta}q_{31} + \dot{\varphi}^2 q_{32} + \dot{\vartheta}^2 q_{33} + \dot{\varphi}\dot{\chi}q_{34} + \dot{\vartheta}\dot{\chi}q_{35} + S_1 q_{36} + S_2 q_{37} + S_3 q_{38} = 0 \qquad (16)$$

$$
\begin{aligned}
q_{31} &= h_{32} + h_{34} + h_{35} - q_{12}h_{31} & q_{35} &= -q_{13}h_{31} \\
q_{32} &= h_{33} - q_{14}h_{31} & q_{36} &= -q_{15}h_{31} \\
q_{33} &= h_{36} & q_{37} &= -q_{16}h_{31} \\
q_{34} &= -q_{11}h_{31} & q_{38} &= -S_3
\end{aligned}
\tag{17}
$$

Eqs.(12),(14), (16) reads

$$\ddot{\varphi} + \dot{\varphi}\dot{\chi}q_{11} + \dot{\varphi}\dot{\vartheta}q_{12} + \dot{\vartheta}\dot{\chi}q_{13} + \dot{\varphi}^2 q_{14} + S_2 q_{15} + S_1 q_{16} = 0 \tag{18}$$

$$\ddot{\vartheta} + \dot{\chi}\dot{\varphi}q_{21} + \dot{\varphi}\dot{\vartheta}q_{22} + \dot{\vartheta}\dot{\chi}q_{23} + \dot{\varphi}^2 q_{24} + S_1 q_{25} + S_2 q_{26} = 0 \tag{19}$$

$$\ddot{\chi} + \dot{\varphi}\dot{\vartheta}q_{31} + \dot{\varphi}^2 q_{32} + \dot{\vartheta}^2 q_{33} + \dot{\varphi}\dot{\chi}q_{34} + \dot{\vartheta}\dot{\chi}q_{35} + S_2 q_{36} + S_1 q_{37} + S_3 q_{38} = 0 \tag{20}$$

$$
\begin{aligned}
q_{ij} &= q_{ij}(\vartheta,\varphi,\chi) \qquad ; \; i,j = 1,2,\ldots\ldots \\
S_i &= S_i(\vartheta,\varphi,\chi) \qquad ; \; i = 1,2,3
\end{aligned}
\tag{21}
$$

Eqs.(18), (19), (20) can be written in the form

$$
\begin{aligned}
\ddot{\varphi} &= f_1(\dot{\varphi},\dot{\vartheta},\dot{\chi},\varphi,\vartheta,\chi) \\
\ddot{\vartheta} &= f_2(\dot{\varphi},\dot{\vartheta},\dot{\chi},\varphi,\vartheta,\chi) \\
\ddot{\chi} &= f_3(\dot{\varphi},\dot{\vartheta},\dot{\chi},\varphi,\vartheta,\chi)
\end{aligned}
\tag{22}
$$

For the forcefree, symmetric gyroscope, i.e., $_3 = 0$, $S_i = 0$ $(i=1,2,3)$ (see Eqs. (7), (9), (11)), Eq.(22) reads

$$
\begin{aligned}
\ddot{\varphi} &= f_{1ffs}(\dot{\varphi},\dot{\vartheta},\dot{\chi},\varphi,\vartheta,\chi) \\
\ddot{\vartheta} &= f_{2ffs}(\dot{\varphi},\dot{\vartheta},\dot{\chi},\varphi,\vartheta,\chi) \\
\ddot{\chi} &= f_{3ffs}(\dot{\varphi},\dot{\vartheta},\dot{\chi},\varphi,\vartheta,\chi),
\end{aligned}
\tag{23}
$$

where f_{iffs} $(i=1,2,3)$ indicates forcefree, symmetric

For the forcefree asymmetric gyroscope i.e., $S_i=0$; $(i=1,2,3)$ (see Eqs.(7), (9), (11)), Eq.(22) reads

$$\begin{aligned}\ddot{\varphi} &= f_{1ffa}(\dot{\varphi},\dot{\vartheta},\dot{\chi},\varphi,\vartheta,\chi)\\ \ddot{\vartheta} &= f_{2ffa}(\dot{\varphi},\dot{\vartheta},\dot{\chi},\varphi,\vartheta,\chi)\\ \ddot{\chi} &= f_{3ffa}(\dot{\varphi},\dot{\vartheta},\dot{\chi},\varphi,\vartheta,\chi),\end{aligned} \tag{24}$$

where f_{iffa} indicates the forcefree, asymmetric case.

For the symmetric heavy gyroscope, i.e., $\mathcal{X}_3=0$ (see Eq.(11)), Eq.(22) reads

$$\begin{aligned}\ddot{\varphi} &= f_{1sh}(\dot{\varphi},\dot{\vartheta},\dot{\chi},\varphi,\vartheta,\chi)\\ \ddot{\vartheta} &= f_{2sh}(\dot{\varphi},\dot{\vartheta},\dot{\chi},\varphi,\vartheta,\chi)\\ \ddot{\chi} &= f_{3sh}(\dot{\varphi},\dot{\vartheta},\dot{\chi},\varphi,\vartheta,\chi)\end{aligned} \tag{25}$$

Eq.(22) can be written in the form

$$\begin{aligned}&\dot{Z}_1 : \dot{\varphi}_1 = \varphi_2 \qquad && \dot{Z}_4 : \dot{\vartheta}_2 = f_2\\ &\dot{Z}_2 : \dot{\varphi}_2 = f_1 && \dot{Z}_5 : \dot{\chi}_1 = \chi_2\\ &\dot{Z}_3 : \dot{\vartheta}_1 = \vartheta_2 && \dot{Z}_6 : \dot{\chi}_2 = f_3\end{aligned} \tag{26}$$

The sign ":" indicates that $\dot{\varphi}_1 \equiv \dot{Z}_1$, e.g.

For domains, where f_i ($i=1,2,3$) are holomorphic the formal solution of Eq. (26) reads /1/,/8/ (see appendix):

$$Z_i = e^{tD} z_i \ , \tag{27}$$

where
$$D = \varphi_2\frac{\partial}{\partial z_1} + f_1\frac{\partial}{\partial z_2} + \vartheta_2\frac{\partial}{\partial z_3} + f_2\frac{\partial}{\partial z_4} + \chi_2\frac{\partial}{\partial z_5} + f_3\frac{\partial}{\partial z_6} \tag{28}$$

A) Representation of the Solution S in the Form $S = S_{\text{symmetric, heavy}}$ + Contributions from Asymmetry.

Starting from Eq.(22) we obtain for the operator D /1/

$$D = \varphi_2\frac{\partial}{\partial z_1} + \vartheta_2\frac{\partial}{\partial z_2} + \chi_2\frac{\partial}{\partial z_3} + f_{1sh}\frac{\partial}{\partial z_4} + f_{2sh}\frac{\partial}{\partial z_5} + f_{3sh}\frac{\partial}{\partial z_6} +$$

$$+ f_{1a}\frac{\partial}{\partial z_2} + f_{2a}\frac{\partial}{\partial z_4} + f_{3a}\frac{\partial}{\partial z_6} , \tag{29}$$

where f_{ia} (i=1,2,3) indicate the contribution from asymmetry and f_{ish} (i=1,2,3) indicates the symmetric, heavy case. We write now the operator D in the form

$$D = D_1 + D_2,$$

where
$$D_2 = f_{1a}\frac{\partial}{\partial z_2} + f_{2a}\frac{\partial}{\partial z_4} + f_{3a}\frac{\partial}{\partial z_6} , \tag{30}$$

and D_1 is defined by (29). The solution Eq.(27) reads in this case

$$Z_i = e^{tD} z_i = e^{t(D_1+D_2)} z_i = e^{tD_1} z_i + \sum_{0}^{\infty} \int_{t_0}^{t} \frac{(t-\tau)^{\alpha}}{\alpha !} \left[D_2 D^{\alpha} z_i\right]_{\bar{a}} d\tau \tag{31}$$

The subscript $\bar{a}$ indicates that after applying $D_2 D$ on z_i, z_i das to be replaced by $e^{tD_1} z_i$. In Eq.(31) the operator D_1 is the operator for the symmetric heavy gyroscope

The solution representation is recommandable, if the deviations from the symmetrc gyroscopr are small. In this case only few terms of the sum in Eq.(31) have to be taken into account. For the evaluation of the integral appearing in Eq.(31) suitable methods are already developed /6/,/7/; in these works also the problem of err estimation is treated.

Moreover we will use another method for solving Eq.(1), as proposed by GROEBNER /2/. For that we put

$$\frac{I_2 - I_1}{I_3} = \varepsilon , \tag{32}$$

where is a parameter. Using this parameter we obtain $f_{ia} = \varepsilon f^{*}_{ia}$ (i=1,2,3) and the operator D reads

$D = D_1 + D_2 = D_1 + \varepsilon D_2^*$, where the operator D_2 reads

$$D_2 = f_{1a}^* \frac{\partial}{\partial z_2} + f_{2a}^* \frac{\partial}{\partial z_4} + f_{3a}^* \frac{\partial}{\partial z_6} , \tag{33}$$

where D_1 and D_2 are given by Eq.(29) and Eq.(3o), respectively. With (27) the solution reads

$$\begin{aligned} Z_i &= e^{t(D_1+D_2)} z_i = e^{t(D_1+\varepsilon D_2^*)} z_i = \sum_{j=0}^{\infty} \varepsilon^j g_j(t,z_i) = \\ &= g_o(t,z_i) + \sum_{j=1}^{\infty} \varepsilon^j g_j , \end{aligned} \tag{34}$$

where $g_o(t,z_i) = e^{tD_1} z_i$ and g_{j+1} can be calculated by the following recurrence formula/2/

$$g_{j+1}(t,z_i) = \int_{t_o}^{t} \left[D_2^* g_j(\tau,z_i) \right]_{z_i \rightarrow g_o(t-\tau,z_i)} d\tau \tag{35}$$

The subscript $z_i \rightarrow g_o(t-\tau,z_i)$ indicates that after applying the operator D_2 on g_j, z_i has to be replaced by $g_o(t-\tau,z_i)$. The proof of formula (34) and formula (35) is given in the work by GROEBNER /2/. D_1 is the operator for the heavy symmetric gyroscope for which a global representation exists. Since the quantity   defined by Eq.(33) is usually small the factor ε^j ($j = 1,2,3....$) influences the convergence in a favorable way.

B) <u>Representation of the Solution S of Eq.(1) in the Form $S = S_{\text{symmetric, forcefree}}$ + Contributions From Asymmetry and Forces</u>.

In this case we write the operator D in the form

$$D = \varphi_2\frac{\partial}{\partial z_1} + \psi_2\frac{\partial}{\partial z_3} + \chi_2\frac{\partial}{\partial z_5} + f_{1sff}\frac{\partial}{\partial z_2} + f_{2sff}\frac{\partial}{\partial z_4} + f_{3sff}\frac{\partial}{\partial z_6} +$$

$$+ f_{1h}\frac{\partial}{\partial_2} + f_{2h}\frac{\partial}{\partial z_4} + f_{3h}\frac{\partial}{\partial z_6} + f_{1a}\frac{\partial}{\partial z_2} + f_{2a}\frac{\partial}{\partial z_4} + f_{3a}\frac{\partial}{\partial z_6}, \tag{36}$$

where f_{ih} $(i=1,2,3)$ indicate the contribution of the external force (heavy), f_{isff} and f_{ia} $(i=1,2,3)$ are explained above. We put now

$$D = D_1 + D_{ha}, \text{ where}$$

$$D_{ah} = f_{1h}\frac{\partial}{\partial z_2} + f_{2h}\frac{\partial}{\partial z_4} + f_{3h}\frac{\partial}{\partial z_6} + f_{1a}\frac{\partial}{\partial z_2} + f_{2a}\frac{\partial}{\partial z_4} + f_{3a}\frac{\partial}{\partial z_6} \tag{37}$$

and D_1 is defined by Eq.(36). Solution (27) reads in our case

$$Z_i = e^{tD} z_i = e^{t(D_1+D_{ha})} z_i = e^{tD_1} z_i + \sum_{o}^{\infty} \int_{t_o}^{t} \frac{(t-\tau)^{\chi}}{\chi!} \left[D_{ha} D^{\chi} z_i\right]_{\bar{b}} d\tau \tag{38}$$

D_1 is the operator for the symmetric forcefree gyroscope, i.e., $e^{tD_1} z_i$ is the solution of the symmetric forcefree gyroscope.

Taking account of the different torque acting on the gyroscope M_1, M_2 and M_3 in Eq.(1) reads

$M_1 = \sum_i M_{1i}$; $M_2 = \sum_i M_{2i}$; $M_3 = \sum_i M_{3i}$, where $i = 1,2,\dots\dots$ (39)

indicates the different torques.

Considering a satellite considerable torques are, e.g.: The gravitational torque, the drag torque, the torque caused by radiation and the magnetic torques. Splitting off the operator $D_{ah} = D_a + D_h$, where

$$D_a = f_{1a}\frac{\partial}{\partial z_2} + f_{2a}\frac{\partial}{\partial z_4} + f_{3a}\frac{\partial}{\partial z_6} \tag{40}$$

$$D_h = f_{1h}\frac{\partial}{\partial z_2} + f_{2h}\frac{\partial}{\partial z_4} + f_{3h}\frac{\partial}{\partial z_6} \tag{41}$$

D_h again can be written in the form $\sum\limits_{l} D_{lh}$ $(l=1,2,3\ldots.) =$ D_h, where

$$D_{lh} = f_{1lh}\frac{\partial}{\partial z_2} + f_{2lh}\frac{\partial}{\partial z_4} + f_{3lh}\frac{\partial}{\partial z_6} \;;\; l=1,2,\ldots. \tag{42}$$

and $f_{11h} = S_{21}q_{25}$; $f_{21h} = S_{11}q_{25} + S_{21}q_{26}$; $f_{31h} = S_{11}q_{37} +$
$+ S_{31}q_{38}$ (43)

Eq.(38) has now the form

$$Z_i = e^{tD_1} z_i + \sum_{o}^{\infty}\int_{t_o}^{t} \frac{(t-\tau)^{\varkappa}}{\varkappa!}\left[D_a^{\varkappa} Dz_i\right]_{\bar{b}} d\tau + \sum_{o}^{\infty}\int_{t_o}^{t} \frac{(t-\tau)^{\varkappa}}{\varkappa!}\left[\sum_{l} D_{hl}^{\varkappa} Dz_i\right]_{\bar{b}} d\tau \tag{44}$$

The subscript $\bar{b}$ indicates, that after applying the operator, z_i has to be replaced by $e^{tD_1} z_i$. The last integral term in Eq.(44) vanishes if M_i $(i=1,2,3)$ is equal to zero. The solution representation Eq.(44) enables us to evaluate the single integral terms numerically independently from the other terms.

C) <u>Representation of the Solution S of Eq.(1) in the Form</u>
$S = S_{\text{asymmetric,forcefree}}$ + <u>Contributions From Forces</u>

In this case we write the operator D in the form

$D = D_{1aff} + D_h$, where the operator D_{1aff} reads

$$D_{1aff} = \varphi_2\frac{\partial}{\partial z_1} + \varphi_2\frac{\partial}{\partial z_3} + \chi_2\frac{\partial}{\partial z_5} + f_{1sff}\frac{\partial}{\partial z_2} + f_{2sff}\frac{\partial}{\partial z_4} + f_{3sff}\frac{\partial}{\partial z_6} +$$

$$+ f_{1a}\frac{\partial}{\partial z_2} + f_{2a}\frac{\partial}{\partial z_4} + f_{3a}\frac{\partial}{\partial z_6} \qquad (45)$$

$$D_h = f_{1h}\frac{\partial}{\partial z_2} + f_{2h}\frac{\partial}{\partial z_4} + f_{3h}\frac{\partial}{\partial z_6} \qquad (46)$$

The solution (27) reads in this case

$$Z_i = e^{tD} z_i = e^{t(D_{1aff}+D_h)} z_i = e^{tD_{1aff}} z_i + \\ + \sum_{o}^{\infty} \int_{t_o}^{t} \frac{(t-\tau)^{\alpha}}{\alpha!} \left[D_h D z_i \right]_{\bar{a}} d\tau \qquad (47)$$

D_{1aff} is the operator for the asymmetric forcefree gyroscope for which a global solution representation exists/3,4,5/.

Putting $e^{tD_{1aff}} z_i = Z_{i1}$ and splitting off the operator D_{1aff} in the form $D_{1aff} = D_{1ff} + \varepsilon \overset{*}{D_2}$, where D_2 is defined by (33) and D_{1ff} is defined by Eq.(45), we obtain the solution in the form

$$Z_{i1} = e^{tD_{1aff}} z_i = e^{t(D_{1ff}+\varepsilon \overset{*}{D_2})} z_i = \sum_{j=o}^{\infty} \varepsilon^j g_j(t,z_i) = \qquad (48)$$

$g_o(t,z_i) + \sum_{j=1}^{\infty} \varepsilon^j g_j(t,z_i)$, where g_o is given by the relation

$$g_o(t,z_i) = e^{tD_{1ff}} z_i \text{ and } g_{j+1}(t,z_i) = \int_{t_o}^{t} \left[\overset{*}{D_2} g_j(\tau,z_i) \right]_{z_i \to g_o(t,z_i)} d\tau \qquad (49)$$

With Eq.(48), Eq.(47) reads

$$Z_i = e^{tD_{1ff}} z_i + \sum_{j=1}^{\infty} \varepsilon^j g_j(t,z_i) + \sum_{o}^{\infty} \int_{t_o}^{t} \frac{(t-\tau)^{\alpha}}{\alpha!} \left[D_h \overset{\alpha}{D} z_i \right]_{\bar{a}} d\tau \qquad (50)$$

where D_{1ff} is the operator for the forcefree symmetric gyroscope.

If several external forces are present we obtain in analogy to Eq.(44) the solution Eq.(5o) in the form

$$Z_i = e^{tD_{1ff}} + \sum_{j=1}^{\infty} \xi^j g_j(t, z_i) + \sum_{o}^{\infty} \int_{t_o}^{t} \frac{(t-\tau)^{\alpha}}{\alpha!} \left[\sum_{l} D_{hl} D^{\alpha} z_i \right]_{\bar{a}} d\tau \tag{51}$$

This representation is advantageous insofar as it contains several additive integral terms, which can be computed separately. The number of summation terms $\alpha = o, 1, 2, \ldots$ depends on the order of magnitude of the torque appaering in the operators D_{hl} ($l = 1, 2, \ldots\ldots$). For the numerical evaluation of the integral terms we refer to the work by H.KNAPP /6,7/, where also the problem of error estimation is treated.

Concerning the stability of the solution of Eq.(1) we refer to the books by KLEIN F. and A. SOMMERFELD/4/ and by R.GRAMMEL /5/, in which this problem is treated in detail.

APPENDIX

Lie Series Method for Solving Differential Equations

Although the Lie series method for solving differential equations is already presented in several publications /1/, /8/, e.g., we will repeat here the essential points of this method. For proofs we refer to the book by GROEBNER /1/.

An ordinary differential equation of order n given by

$$Z^{(n)}(t) = f(t,\ Z,\ Z',\ Z'',\ \ldots\ldots,\ Z^{(n-1)}) \qquad \text{(A-1)}$$

can be written in the form

$$\begin{aligned} Z_1' &= f_1(t, Z_1,\ Z_2,\ \ldots,\ Z_n) \\ Z_2' &= f_2(t, Z_1,\ Z_2,\ \ldots,\ Z_n) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ Z_n' &= f_n(t, Z_1,\ Z_2,\ \ldots,\ Z_n) \end{aligned} \qquad \text{(A-2)}$$

Assuming the functions $f_i(i=1,2,\ldots\ldots,n)$ to be analytic the formal solution of Eq.(A-2) reads

$$Z_i = \sum_{o}^{\infty} \frac{t^\alpha}{\alpha!} D^\alpha z_i \qquad \text{(A-3)}$$

$$(i=1,2,\ldots\ldots,n)$$

The small letter z_i in (A-3) indicates, that after applying the operator D the functions z_i $(i=1,2,\ldots\ldots,n)$ have to be replaced by initial values. The operator D is defined by the relation

$$D = \sum_{i=o}^{\infty} \frac{\partial}{\partial z_i}, \text{ and } {}_i \text{ is defined by} \qquad \text{(A-4)}$$

$$\mathfrak{z}_i = Z_i' \text{ and } \mathfrak{z}_o' = Z_o' = t' = 1 \qquad \text{(A-5)}$$

Eq.(A-6), however, is only valid for non-autonomous systems i.e., for systems where the functions $f_i (i=1,2,\ldots,n)$ contain the linear independent variable t explicitely, otherwise $\mathfrak{J}_0=0$.

As an example we will solve Eq.(26). This equation reads

$$\begin{aligned} \dot{Z}_1 &= \varphi_2 & \qquad \dot{Z}_4 &= f_2 \\ \dot{Z}_2 &= f_1 & \qquad \dot{Z}_5 &= \chi_2 \\ \dot{Z}_3 &= \mathfrak{J}_2 & \qquad \dot{Z}_6 &= f_3 \end{aligned} \tag{A-6}$$

The solution reads

$$Z_i = e^{tD} z_i = \sum_0^\infty \frac{t^\alpha}{\alpha!} D^\alpha z_i , \tag{A-7}$$

where the operator D is according to (A-4) and (A-5) given by the relation

$$D = \varphi_2 \frac{\partial}{\partial z_1} + f_1 \frac{\partial}{\partial z_2} + \mathfrak{J}_2 \frac{\partial}{\partial z_3} + f_2 \frac{\partial}{\partial z_4} + \chi_2 \frac{\partial}{\partial z_5} + f_3 \frac{\partial}{\partial z_6} \tag{A-8}$$

APPENDIX I

In the book, "The General Problem of the Motion of Coupled Rigid Bodies About a Fixed Point", by E. Leimanis, Springer Tracts in Natural Philosophy, Vol.7, 1965, p 133, the Euler Poisson equations of motion are solved by Lie Series. These equations read

$$\begin{aligned} I_1\dot{\Omega}_1 + (I_2 - I_3)\Omega_2\Omega_3 &= mg(\beta z_o - \gamma y_o) \\ I_2\dot{\Omega}_2 + (I_1 - I_3)\Omega_1\Omega_3 &= mg(\gamma x_o - \alpha z_o) \\ I_3\dot{\Omega}_3 + (I_2 - I_1)\Omega_1\Omega_3 &= mg(\alpha y_o - \beta x_o) \end{aligned} \qquad \text{(AI-1)}$$

$$\begin{aligned} \dot{\alpha} &= \beta\Omega_3 - \gamma\Omega_2 \\ \dot{\beta} &= \gamma\Omega_1 - \alpha\Omega_3 \\ \dot{\gamma} &= \alpha\Omega_2 - \beta\Omega_1 \quad , \end{aligned} \qquad \text{(AI-2)}$$

where I_i are the moments of inertia, Ω_i are the angular velocities, 1,2,3 indicate the axis fixed with respect to the body, m is the mass of the body, gm is the weight of the body, $\underline{r}_o = (x_o, y_o, z_o)$ indicates the position of the mass center. (x,y,z) denote the reference frame fixed with respect to the body, α, , are the direction cosines of a fixed axis (Z axis of a space fixed reference frame, e.g.) with respect to x,y,z. E. Leimanis represents the solution of Eqs.(AI-1) and (AI-2) in the form $Z_i = e^{tD} z_i$, where $Z_1 = \Omega_1$, $Z_2 = \Omega_2$, $Z_3 = \Omega_3$, $Z_4 = \alpha$, $Z_5 = \beta$, $Z_6 = \gamma$ and the Lie operator D reads

$$D = \left[\frac{mg}{I_1}(\beta z_o - \gamma y_o) - \frac{I_2 - I_3}{I_1}\Omega_2\Omega_3\right]\frac{\partial}{\partial\Omega_1} + \frac{mg}{I_2}\left[\gamma x_o - \alpha z_o) - \frac{I_1 - I_3}{I_2}\Omega_1\Omega_3\right]\frac{\partial}{\partial\Omega_2} + \frac{mg}{I_3}\left[\alpha y_o - \beta x_o) - \frac{I_2 - I_1}{I_3}\Omega_1\Omega_3\right]\frac{\partial}{\partial\Omega_3} +$$

As shown in this paper we have solved the Euler equation containing not only a term for the gravitational torque, but also several other terms corresponding to other torques (drag torque, centrifugal torque, e.t.c.). Furthermore, as far as the solution representation is concerned experience /6/ has shown, that a representation as it was given by E. Leimanis is not recommendable for numerical computation. A rearrangement of the series $e^{tD}z_i$ by splitting off in the form $e^{t(D_1+D_2)}z_i = e^{tD_1}z_i + R$, as it was done in this report, influences the numerical evaluation in a favorable way, if, e.g., $e^{tD_1}z_i \gg R$.

Referring to the solution representation presented in this report (see Eqs. (31), (38), (44), (47)) we emphasize that the numerical usefulness of these representations depends on the problem considered. The effectiveness of the aforementioned method depends highly on the skill to rearrange the series.

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